Introduction to $(\infty,1)$- categories

<u>Idea</u>: - introduce "homotopy" to categories:

have seen examples: (Top, weak hom equiv)

($Ch_{\mathbb{K}}$, quasi-isos)

want to "invert" these

- "∞" have k-morphisms $\forall k$,

composition associative up to higher morphisms

↳ objects

(1-)morphisms b/w objects $\bullet \longrightarrow \bullet$

2-morphisms b/w 1-morphisms

3-morphisms b/w 2-morphisms

⋮

- "(∞,n)": $\forall k > n$, k-morphisms are invertible

<u>Ex</u>: $Ch_{\mathbb{K}}$: obj. chain complexes

1-mor: quasi-isom

2-mor: inv. homotopies of quasi-isos

3-mor: inv homotopies of ↑

are invertible up to a higher mor

<u>Question</u>: How to make these ideas into a formal definition?

<u>non-answer</u>:
- There is not one definition, but several possible ones! ("models")
- BUT, are all equivalent!
- different models have different (dis-) advantages!

Goal for today: See several models, starting with a rather intuitive one

tomorrow: Stefano will focus on one ("quasi-categories") for which ~~[illegible]~~ a good machinery for categorical constructions (adjoints, limits, ...) has been developped in great detail.

$\boxed{(\infty,0)\text{-categories} = \infty\text{-groupoids}}$

$$\mathrm{Top} \ni X \longmapsto \pi_{\leq 1} X$$

fundamental groupoid:
- obj: pts in X
- mor: paths in X

$$X \longmapsto \pi_{\leq n} X$$

n-groupoid:
- 2-mor: homotopies of paths
- 3-mor: hom. of hom.
- ⋮
- n-mor

Rem: for small n, get equivalence (for which there are defs of RHS)

$$\text{homotopy } n\text{-type} \xleftrightarrow{\pi_{\leq n}} n\text{-groupoid}$$

... go all the way to ∞:

$$X \longmapsto \pi_{\leq \infty} X$$

fundamental ∞-groupoid
- ⋮
- $(n+1)$-mor
- ⋮

→ Make this into a definition:

Def'n: An <u>∞-groupoid</u> is a space.

Rem: By "space", we mean a $\boxed{\text{model}}$ category of spaces

"model"

we have seen eg $\mathrm{Top} \underset{\text{equiv.}}{\overset{\text{Quillen}}{\rightleftarrows}} \mathrm{sSet}$

$(\infty,1)$-categories

<u>Idea:</u> • (small) category = category enriched in Set
(internal to)

• for any two objects x,y in an $(\infty,1)$-category $\mathcal{C}$, $\mathcal{C}(x,y)$ should be an $(\infty,0)$-category, i.e., a space.

<u>1st model</u>: categories enriched over spaces
(internal to)

<u>terminology</u>: "space" = Top : "topologically enriched categories"
sSet : "simplicial(ly enriched) categories"

<u>Def'n</u> A category enriched over spaces consists of

- set $\mathrm{Ob}\,\mathcal{C}$ of objects
- $\forall a,b \in \mathrm{Ob}\,\mathcal{C}$, a space $\mathcal{C}(a,b)$ of morphisms
- composition maps $\forall a,b,c \in \mathrm{Ob}\,\mathcal{C}$

$$\mathcal{C}(b,c) \times \mathcal{C}(a,b) \longrightarrow \mathcal{C}(a,c)$$

which are | continuous
| maps of sSet.

and which are associative.

<u>Example</u>: (A) <u>sSet</u> is a simplicial category whose objects are simplicial sets, and the space of morphisms is given by the internal Hom:

$\underline{\mathrm{sSet}}(X,Y)_n = \mathrm{sSet}(X \times \Delta[n], Y)$, face + degeneracies from those of $(\Delta[n])$

Example (B) $\underline{Ch}_R$, R comm. ring, is the simplicial category of chain complexes. Mapping spaces arise as follows:

1: $$\mathrm{Hom}_{\underline{Ch}_R}(X,Y)_n = \prod_i \mathrm{Hom}_{Ch_R}(X_i, Y_{i+n})$$

$$\Big\downarrow\; d(f)_i = d \circ f_i - (-1)^n f_{i-1} \circ d$$

$$\mathrm{Hom}_{\underline{Ch}_R}(X,Y)_{n-1}$$

is a chain complex (namely, the internal Hom).

2: This chain complex can be truncated

3 Now use Dold-Kan correspondence.

Def'n
Homotopy category $\mathrm{Ho}\,\mathcal{C}$ of a cat. enriched in spaces $\mathcal{C}$

$$\mathrm{ob}\,\mathrm{Ho}\,\mathcal{C} = \mathrm{ob}\,\mathcal{C}$$

$$\mathrm{mor}_{\mathrm{Ho}\mathcal{C}}(x,y) = \pi_0\,\mathcal{C}(x,y)$$

Bergner: There is a model category of simplicially enriched categories. We denote it by Cat_Δ.

Example © Let (M, W) be a model category, or more generally, a category with weak equivalences $(\mathcal{C}, W)$ (or even more generally, a "relative category")

2nd model

We get a simplicial category by

- simplicial localization
- $\boxed{\text{hammock} \quad \text{—''—}}$

give equivalent simpl. cats

gives enhancement of $\mathcal{C}[W^{-1}] = \mathrm{Ho}\,\mathcal{C}$

Recall: $\mathrm{ob}\ \mathrm{Ho}\,\mathcal{C} = \mathrm{ob}\,\mathcal{C} \quad \left(\simeq \mathcal{C}^{cf}/_\sim\right)$

mor $\mathrm{Ho}\,\mathcal{C}$ are equivalence classes of zig-zags

$$X_0 \xleftarrow{\sim} X_1 \to X_2 \xleftarrow{\sim} \cdots \to X_n,$$

where $\to$ are morphisms in $\mathcal{C}$

$\xleftarrow{\sim}$ —''— W,

so we are "formally inverting morph. in W"

Dwyer-Kan: hammock localization $L^H(\mathcal{C}, W)$ is the simplicial category defined as follows:

⊛ $\mathrm{ob}\ L^H(\mathcal{C}, W) = \mathrm{ob}\,\mathcal{C}$

⊛ Mapping spaces are constructed by: for $X, Y \in \mathrm{ob}\,\mathcal{C}$

1. Consider the following category for fixed n

$$\mathrm{ob} = X \xleftarrow{\sim} X_1 \to X_2 \xleftarrow{\sim} \cdots \xleftrightarrow{\sim} X_{n-1} \to Y$$

mor =

$$\begin{tikzcd}
 & X_1 \arrow[dl, "\sim"'] \arrow[r] \arrow[d, "\sim"] & X_2 \arrow[d, "\sim"] & \cdots \arrow[l, "\sim"'] & X_{n-1} \arrow[l, "\sim"'] \arrow[d, "\sim"] \arrow[dr] & \\
X & \tilde{X}_1 \arrow[l, "\sim"] \arrow[r] & \tilde{X}_2 & \cdots \arrow[l, "\sim"'] & \tilde{X}_{n-1} \arrow[l, "\sim"'] \arrow[r] & Y
\end{tikzcd}$$

so, the data of weak equiv $X_1 \xrightarrow{\sim} \tilde{X}_1$, ..., $X_{n-1} \xrightarrow{\sim} \tilde{X}_{n-1}$ st the diagram commutes.

Take the nerve of this category to get a sSet.

2. Take coproduct over all n (i.e. put all these together)

3. mod out by equivalence relation given by
   - inserting/deleting identities
   - composing morphisms.

(*) Composition of mapping spaces is given by concatenating chains.

Rem:

1) $\mathrm{Ho}\, L^H(\mathcal{C}, W) \simeq \mathcal{C}[W^{-1}]$

2) If we started w/ a (simplicial) model category, the extra data allows to compute these mapping spaces using (co)fibrant resolutions:

$$L^H\mathcal{C}(a,b) = \mathcal{C}(\mathrm{cof}(a), \mathrm{fib}(b))$$

3) For the right notion of "equivalence" (= DK-equiv, see below) Quillen equivalences give equivalent simpl. cat.

More generally, $\mathrm{RelCat} \overset{L^H}{\dashrightarrow} \mathrm{Cat}_\Delta$ equiv. in suitable sense.

model cat of "relative cats" model cat of simpl. cat

Def'n: A map $F: \mathcal{C} \to \mathcal{D}$ of simplicial categories is a Dwyer-Kan equivalence if

- it induces weak homotopy equivalences of the mapping spaces
$$\mathrm{Map}_{\mathcal{C}}(a,b) \to \mathrm{Map}_{\mathcal{D}}(Fa, Fb)$$
- it induces an equivalence of the homotopy categories
$$\mathrm{Ho}(F): \mathrm{Ho}\,\mathcal{C} \xrightarrow{\sim} \mathrm{Ho}\,\mathcal{D}$$

3rd model: (complete) Segal spaces

- associativity "on the nose" might be too much
- Having a unique composition can be weakened to having a contractible space of comp.
(Example: bordism categories)

Def'n: A simplicial space $X_\bullet: \Delta^{op} \to \text{Spaces}$ is a Segal space if:

$\forall\ 0 < i \leq n$, there are maps in Δ:
$$\gamma_i: [1] \to [n]$$
$$\begin{cases} 0 \mapsto i-1 \\ 1 \mapsto i \end{cases}$$
these induce maps $X_n \xrightarrow{\gamma_i} X_1$, which fit together to give
$$X_n \xrightarrow{(\gamma_i)} X_1 \overset{h}{\underset{X_0}{\times}} \cdots \overset{h}{\underset{X_0}{\times}} X_1$$

We require this map to be a weak equivalence:
$$X_n \xrightarrow[\sim]{(\gamma_i)} X_1 \overset{h}{\underset{X_0}{\times}} \cdots \overset{h}{\underset{X_0}{\times}} X_1$$

Think of .... X_0 as the space of objects

X_1 as the space of all morphisms

$X_1 \rightrightarrows X_0$ as the source and target maps

cf: $\mathcal{C}$ category.

$$(N\mathcal{C})_n = \underbrace{\coprod_{x_0,\dots,x_n}}_{\text{this is weakened for spaces.}} \underbrace{\mathcal{C}(x_n,x_{n-1})}_{\text{sets}} \times \mathcal{C}(x_{n-1},x_{n-2}) \times \dots \times \mathcal{C}(x_0,x_1)$$

We won't discuss completeness here; it ensures that the $(\infty,1)$-cat interpretation of a Segal space is not ambiguous (cf. Lurie's paper on the Cobordism Hypothesis) Explanation in

Rezk: There is a model category describing complete Segal spaces, denote it by CSS

Barwick-Kan: defined nerve of relative categories, (after Rezk) which gives a Segal space

Quillen adjunction

$$\mathrm{RelCat} \underset{N}{\overset{\longleftarrow}{\longrightarrow}} \mathrm{CSS}$$

$$\mathrm{CSS} \rightleftarrows \vdots \rightleftarrows \mathrm{Cat}_\Delta$$

← sequence of Quillen adjunctions

Toën: Can axiomatize $(\infty,1)$-cats. Up to action of $\mathbb{Z}/2\mathbb{Z}$ corresponding to taking the opposite category, they are unique.